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Invariant Imbedding and the Resolvent of Fredholm Integral Equations with Semi-Degenerate Kernels*

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In many branches of astrophysics, physics, biology, and nuclear engineering, the underlying functional equation is a Fredholm integral equation. In this paper, it is shown that Fredholm integral equations with semi-degenerate kernels can be reduced to initial-value problems for systems of ordinary differential equations using an interesting formula for the Fredholm resolvent. Semi-degenerate kernels are encountered in many applications in the foregoing fields. This procedure facilitates the computational solution of the two-point boundary-value problem by both analog and digital computers.

1. INTRODUCTION

In our preceding papers (cf. Bellman and Ueno, [1] and [2]), the reduction of Fredholm integral equations with asymmetric and composite symmetric kernels to an initial-value problem has been discussed, with the aid of a well-known formula for the resolvent (cf. Bellman [5] and Krein [6]). It is shown there that the Cauchy systems obtained in this fashion are suitable for computational solution by high-speed computers. Recently, an initial-value method has been developed for the solution of a Fredholm integral equation with semi-degenerate kernel, taking into account superposition principles (cf. Kalaba and Vereeke [3]; Kagiwada and Kalaba [4]).

In this paper, we show how to obtain Cauchy systems for the resolvent and the related auxiliary functions for Fredholm integral equation with semi-

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degenerate kernel, with the aid of the foregoing formula. In the present case the resolvent kernel is not symmetric with respect to the relevant depth arguments t and y because the kernel $k(t, y)$ is represented by one degenerate function for $y > t$, and another degenerate function for $y < t$. It is shown that, even in this case, the application of the cited formula for the resolvent enables us to reduce the two-point boundary-value problem to an initial-value problem. This is important because the Cauchy systems for the auxiliary functions are readily computable by means of modern computers.

It may be of interest to mention that the invariant imbedding procedure covers the important cases in which the kernel is of the Green's-function type (cf. Brysk [7]) and also that in which the integral equation is a Volterra type. Finally, it should be mentioned that, although the present problem has been treated by other authors (cf. Kalaba and Vereeke [3]; Kagiwada and Kalaba [4]) with the aid of a initial-value method, their Cauchy systems for the auxiliary functions are different from ours, because we have used a formula for the resolvent instead of the superposition principle. In a subsequent paper, we shall deal with the application to the Volterra integral equation of the second kind.

2. Fredholm Integral Equation and the Resolvent

Consider the Fredholm integral equation

$$u(t, x) = g(t) + \int_0^x k(t, y)\, u(y, x)\, dy, \tag{1}$$

where $0 \leqslant t \leqslant x$, $g(t)$ is a given forcing function, and the kernel has the form

$$k(t, y) = \sum_{i=1}^{M} a(i, t)\, b(i, y), \qquad t > y, \tag{2}$$

$$k(t, y) = \sum_{j=1}^{N} c(j, t)\, d(j, y), \qquad t < y. \tag{3}$$

Under reasonable assumptions, Eq. (1) has a unique solution for sufficiently small value x. An initial-value problem for u is now discussed formally.

Introducing the resolvent kernel $K(t, y, x)$, assumed to exist, the function u is expressed in the form

$$u(t, x) = g(t) + \int_0^x K(t, z, x)\, g(z)\, dz. \tag{4}$$

Integral equations governing the resolvent are

$$K(t, y, x) = k(t, y) + \int_0^x K(t, z, x)\, k(z, y)\, dz, \tag{5}$$

and

$$K(t, y, x) = k(t, y) + \int_0^x k(t, z)\, K(z, y, x)\, dz, \tag{6}$$

where the resolvent K is not symmetric with respect to t and y, because of the semi-degenerate kernel. Upon differentiation of Eq. (5) with respect to x, we have

$$K_x(t, y, x) = K(t, x, x)\, k(x, y) + \int_0^x K_x(t, z, x)\, k(z, y)\, dz, \tag{7}$$

where the subscript represents the partial differentiation with respect to x. Combination of Eqs. (6) and (7) leads to

$$K_x(t, y, x) = K(t, x, x)\, K(x, y, x), \tag{8}$$

where $0 \leqslant t, y \leqslant x$, and $t \geqslant y$. Equation (8) is the Bellman–Krein formula for the resolvent mentioned above (cf. Bellman [5] and Krein [6]). Once $K(t, x, x)$ and $K(x, y, x)$ have been given, then we can numerically compute the resolvent by using this formula (8). With the aid of Eq. (4), we are able to determine the required function $u(t, x)$.

3. Invariant Imbedding for the Auxiliary Equations

Introduce an auxiliary equation (cf. Kalaba and Vereeke [3]; Kagiwada and Kalaba [4]) given by

$$J(j; t, x) = c(j, t) + \int_0^x k(t, y)\, J(j; y, x)\, dy, \tag{9}$$

where $j = 1, 2,..., N$, and $0 \leqslant t \leqslant x$.

Recalling Eq. (6) for $y = x$, we have

$$\begin{aligned} K(t, x, x) &= k(t, x) + \int_0^x k(t, z)\, K(z, x, x)\, dz \\ &= \sum_{j=1}^{N} c(j, t)\, d(j, x) + \int_0^x k(t, z)\, K(z, x, x)\, dz. \end{aligned} \tag{10}$$

Combination of Eqs. (9) and (10) results in

$$K(t, x, x) = \sum_{j=1}^{N} J(j; t, x)\, d(j, x). \tag{11}$$

In a manner similar to Eq. (10), we write

$$\begin{aligned} K(x, y, x) &= k(x, y) + \int_0^x K(x, z, x)\, k(z, y)\, dz \\ &= \sum_{i=1}^{M} a(i, x)\, b(i, y) + \int_0^x k(z, y)\, K(x, z, x)\, dz. \end{aligned} \tag{12}$$

Putting

$$\bar{J}(i; y, x) = b(i, y) + \int_0^x k(z, y)\, \bar{J}(i; z, x)\, dz, \tag{13}$$

where $i = 1, 2, \ldots, M$, $0 \leqslant t \leqslant x$, and combining it with Eq. (12), we get

$$K(x, y, x) = \sum_{i=1}^{M} a(i, x)\, \bar{J}(i; y, x). \tag{14}$$

In what follows, we shall determine J and $\bar{J}$ functions as solutions of an initial-value problem, and then evaluate $K(t, x, x)$ and $K(x, y, x)$ by Eqs. (11) and (14), respectively.

On expressing Eq. (9) in terms of the resolvent, we obtain

$$J(j; t, x) = c(j, t) + \int_0^x K(t, z, x)\, c(j, z)\, dz. \tag{15}$$

Differentiating Eq. (15) with respect to x, we have

$$J_x(j; t, x) = K(t, x, x)\, c(j, x) + \int_0^x K_x(t, z, x)\, c(j, z)\, dz. \tag{16}$$

The insertion of Eq. (8) into Eq. (16) results in

$$\begin{aligned} J_x(j; t, x) &= K(t, x, x) \left[c(j, x) + \int_0^x K(x, z, x)\, c(j, z)\, dz \right] \\ &= K(t, x, x)\, J(j; x, x) \\ &= J(j; x, x) \sum_{n=1}^{N} J(n; t, x)\, d(n, x), \end{aligned} \tag{17}$$

allowing for Eq. (15).

Interchanging t by $x - t$ in Eq. (15), we get

$$J(j; x - t, x) = c(j, x - t) + \int_0^x K(x - t, z, x)\, c(j, z)\, dz. \tag{18}$$

Differentiation of Eq. (18) with respect to x provides us with

$$\begin{aligned} J_x(j; x-t, x) &= c_x(j, x-t) + K(x-t, x, x)\, c(j, x) \\ &\quad + \int_0^x K_x(x-t, z, x)\, c(j, z)\, dz. \end{aligned} \tag{19}$$

On making use of Eq. (8), Eq. (19) leads to [see Eq. (48) for $\bar{c}(j, x-t)$]

$$\begin{aligned} &J_x(j; x-t, x) \\ &\quad = \bar{c}(j, x-t) + K(x-t, x, x)\left[c(j, x) + \int_0^x K(x, z, x)\, c(j, z)\, dz\right] \\ &\quad = \bar{c}(j, x-t) + J(j; x, x) \sum_{n=1}^{N} J(n; x-t, x)\, d(n, x). \end{aligned} \tag{20}$$

Putting $t = 0$ in Eqs. (17) and (20), we obtain the required Cauchy system for the auxiliary functions $J(j; 0, x)$ and $J(j; x, x)$, [see Eq. (53) for $\bar{c}(j, x)$]

$$J_x(j; 0, x) = J(j; x, x) \sum_{n=1}^{N} J(n; 0, x)\, d(n, x), \tag{21}$$

$$J_x(j; x, x) = \bar{c}(j, x) + J(j; x, x) \sum_{n=1}^{N} J(n; x, x)\, d(n, x), \tag{22}$$

where $j = 1, 2,..., N$, together with the initial conditions

$$[J(j; 0, x)]_{x=0} = c(j, 0) \qquad \text{and} \qquad [J(j; x, x)]_{x=0} = c(j, 0). \tag{23}$$

For sufficiently small values of x, Eqs. (22) and (29) determine uniquely the functions $J(j; x, x)$ and $\bar{J}(i; x, x)$.

In what follows, in a manner similar to the above procedure, we shall show how to determine the functions $\bar{J}(i; 0, x)$ and $\bar{J}(i; x, x)$, which enable the auxiliary function $\bar{J}(i, y, x)$ $(0 \leqslant y \leqslant x, i = 1, 2,..., M)$ to be computed. On expressing Eq. (13) in terms of the resolvent, we have

$$\bar{J}(i; y, x) = b(i, y) + \int_0^x K(z, y, x)\, b(i, z)\, dz. \tag{24}$$

On differentiating it with respect to x, and using Eq. (8), we get

$$\begin{aligned} \bar{J}_x(i; y, x) &= K(x, y, x)\, \bar{J}(i; x, x) \\ &= \bar{J}(i; x, x) \sum_{m=1}^{M} a(m, x)\, \bar{J}(m; y, x). \end{aligned} \tag{25}$$

Interchanging y and $x - y$ in Eq. (24), we have

$$\bar{J}(i; x - y, x) = b(i, x - y) + \int_0^x K(z, x - y, x)\, b(i, z)\, dz. \tag{26}$$

Differentiation of Eq. (26) with respect to x results in [see Eq. (54) for $\bar{b}(i, x - y)$]

$$\begin{aligned} \bar{J}_x(i; x - y, x) &= \bar{b}(i, x - y) + K(x, x - y,)\, \bar{J}(i; x, x) \\ &= \bar{b}(i, x - y) + \bar{J}(i; x, x) \sum_{m=1}^{M} a(m, x)\, \bar{J}(m; x - y, x). \end{aligned} \tag{27}$$

Putting $y = 0$ in Eqs. (25) and (27), we obtain the required Cauchy system for $\bar{J}(i; 0, x)$ and $\bar{J}(i; x, x)$ by using the relations [see Eq. (55) for $\bar{b}(i, x)$]

$$\bar{J}_x(i; 0, x) = \bar{J}(i; x, x) \sum_{m=1}^{M} a(m, x)\, \bar{J}(m; 0, x) \tag{28}$$

and

$$\bar{J}_x(i; x, x) = \bar{b}(i, x) + \bar{J}(i; x, x) \sum_{m=1}^{M} a(m, x)\, \bar{J}(m; x, x), \tag{29}$$

together with the initial conditions

$$[\bar{J}(i; 0, x)]_{x=0} = b(i, 0), \qquad [\bar{J}(i; x, x)]_{x=0} = b(i, 0), \tag{30}$$

where $i = 1, 2, \ldots, M$.

Hence, once $J(j; x, x)$ and $\bar{J}(i; x, x)$ have been given by solving Eqs. (21), (22), (28), and (29), Eqs. (17) and (25) determine the required auxiliary functions $J(j; t, x)$ and $\bar{J}(i; y, x)$ for $0 \leqslant t,\ y \leqslant x$, when x is sufficiently small. The Cauchy systems are to be solved subject to the initial conditions

$$J(j; t, t) = J(j; x, x) \qquad \text{at} \quad x = t, \tag{31}$$

$$\bar{J}(i; y, y) = \bar{J}(i; x, x) \qquad \text{at} \quad x = y, \tag{32}$$

where $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$.

Then, with the aid of Eqs. (11) and (14), we can determine the resolvent, i.e., $K(t, x, x)$ and $K(x, y, x)$. The use of Eq. (8) permits us to compute the required resolvent of the Fredholm integral-equation with semi-degenerate kernel.

4. Statement of the Cauchy System

Let us restate the initial-value problems that determine the set of differential equations for the computation of a pair of auxiliary functions $J(i; t, x)$ and $\bar{J}(j; y, x)$ and the resolvent $K(t, y, x)$. The Cauchy system for these functions are as follows: Assume that x_1 is sufficiently small, and $0 \leqslant t, y \leqslant x < x_1$:

$$J_x(j; 0, x) = J(j; x, x) \sum_{n=1}^{N} J(n; 0, x)\, d(n, x), \tag{33}$$

$$J_x(j; x, x) = \bar{c}(j, x) + J(j; x, x) \sum_{n=1}^{N} J(n; x, x)\, d(n, x), \tag{34}$$

where $j = 1, 2,..., N$, together with the initial conditions [see Eq. (53) for $\bar{c}(j, x)$]

$$[J(j; 0, x)]_{x=0} = c(j, 0) \qquad \text{and} \qquad [J(j; x, x)]_{x=0} = c(j, 0). \tag{35}$$

Similarly,

$$\bar{J}_x(i; 0, x) = \bar{J}(i, x, x) \sum_{m=1}^{M} a(m, x)\, \bar{J}(m; 0, x), \tag{36}$$

$$\bar{J}_x(i; x, x) = \bar{b}(i, x) + \bar{J}(i; x, x) \sum_{m=1}^{M} a(m, x)\, \bar{J}(m; x, x), \tag{37}$$

where $i = 1, 2,..., M$, together with the initial conditions [see Eq. (55) for $\bar{b}(i,x)$]

$$[\bar{J}(i; 0, x)]_{x=0} = b(i, 0) \qquad \text{and} \qquad [\bar{J}(i; x, x)]_{x=0} = b(i, 0). \tag{38}$$

The differential equations for $J(j; t, x)$ and $\bar{J}(i; y, x)$ functions are as follows:

$$J_x(j; t, x) = J(j; x, x) \sum_{n=1}^{N} J(n; t, x)\, d(n, x) \tag{39}$$

and

$$\bar{J}_x(i; y, x) = \bar{J}(i; x, x) \sum_{m=1}^{M} a(m, x)\, \bar{J}(m; y, x), \tag{40}$$

together with the initial conditions

$$J(j; t, t) = J(j; x, x) \qquad \text{at} \qquad x = t, \tag{41}$$

and

$$\bar{J}(i; y, y) = \bar{J}(i; x, y) \qquad \text{at} \qquad x = y, \tag{42}$$

where $i = 1, 2,..., M$ and $j = 1, 2,..., N$.

The crucial formula determining the resolvent $K(t, y, x)$ is given by

$$K_x(t, y, x) = K(t, x, x)\, K(x, y, x), \tag{43}$$

where

$$K(t, x, x) = \sum_{j=1}^{N} J(j; t, x)\, d(j, x) \tag{44}$$

and

$$K(x, y, x) = \sum_{i=1}^{M} a(i, x)\, \bar{J}(i; y, x). \tag{45}$$

together with initial conditions by $K(t, y, y)(t \leqslant y)$ and $K(t, y, t)(t \geqslant y)$.

Except for Eqs. (39), (43), and (44), the other equations cited here are new and appear to be useful for the numerical computation of the required quantities.

APPENDIX: Derivation of $\bar{c}(j, x)$ and $\bar{b}(i, x)$-Functions

On recalling Eq. (5), an integral equation governing the resolvent $K(x - t, z, x)$ is expressed in the form

$$K(x - t, z, x) = k(x - t, z) + \int_0^x k(x - t, y)\, K(y, z, x)\, dy. \tag{46}$$

On differentiating with respect to x, and making use of Eq. (8), we have

$$\begin{aligned} K_x(x - t, z, x) &= k_x(x - t, z) + \int_0^x k_x(x - t, y)\, K(y, z, x)\, dy \\ &\quad + k(x - t, x)\, K(x, z, X) + \int_0^x k(x - t, y)\, K_x(y, z, x)\, dy \\ &= k_x(x - t, z) + \int_0^x k_x(x - t, y)\, K(y, z, x)\, dy \\ &\quad + K(x - t, x, x)\, K(x, z, x). \end{aligned} \tag{47}$$

Inserting Eq. (47) into Eq. (19), we get Eq. (20), where $\bar{c}\,(j, x - t)$ is given by

$$\bar{c}(j, x - t) = c_x(j, x - t) + \int_0^x k_x(x - t, z)\, J(j; z, x)\, dz. \tag{48}$$

In Eq. (20), in the limit as $t = 0$, recalling Eq. (2), we obtain

$$\bar{c}(j, x) = c_x(j, x) + \sum_{m=1}^{M} a_x(m, x)\, R(m, j; x), \tag{49}$$

where $j = 1, 2, \ldots, N$, and

$$R(m, j; x) = \int_0^x b(m, z)\, J(j; z, x)\, dz. \tag{50}$$

Upon differentiation of Eq. (49) with respect to x, and making use of Eqs. (17) and (24), we have

$$\begin{aligned} R_x(m, j; x) &= b(m, x)\, J(j; x, x) + \int_0^x b(m, z)\, J_x(j; z, x)\, dz \\ &= J(j; x, x)\, \bar{J}(m; x, x), \end{aligned} \tag{51}$$

together with the initial condition at $x = 0$

$$R(m, j; 0) = 0. \tag{52}$$

Then, allowing for an initial condition given by Eq. (52), combination of Eqs. (49) and (51) results in

$$\bar{c}(j, x) = c_x(j, x) + \sum_{m=1}^{M} a_x(m, x) \int_0^x \bar{J}(m; z, z)\, J(j; z, z)\, dz. \tag{53}$$

where $j = 1, 2, \ldots, N$. Similarly, for $\bar{b}(i, x - y)$ and $\bar{b}(i, x)$ we have

$$\bar{b}(i, x - y) = b_x(i, x - y) + \int_0^x k_x(z, x - y)\, \bar{J}(i; z, x)\, dz, \tag{54}$$

and

$$\bar{b}(i, x) = b_x(i, x) + \sum_{n=1}^{N} d_x(n, x) \int_0^x J(n; z, z)\, \bar{J}(i; z, z)\, dz, \tag{55}$$

where $i = 1, 2, \ldots, M$.

Note added in proof. After the present paper was prepared, we learned that the same problem has been independently treated with the aid of an initial value method by H. Kagiwada and R. Kalaba, in press in the "Proceedings of the 1972 SIAM Symposium on Integral Equations." One of the authors (S.U.) expresses his gratitude to Professor R. Kalaba, University of Southern California, for his helpful discussions on the improvement of the analytical treatment of Eqs. (53) and (55). It is of interest to mention the characteristics of these methods. On recalling Eqs. (9), (13), and (51), after some minor rearrangement of terms, we get a nonlinear differential equation for $R(i, j; x)$, together with the initial condition $R(i, j; 0) = 0$. In a paper mentioned above a Cauchy system for R-function is a basic nonlinear differential equation. However, in the present paper a system of combined integrodifferential equations of Volterra type for $J(i; x, x)$ and $\bar{J}(j; x, x)$ is to be simultaneously solved. The relationship between these equations is similar in form to that between the Cauchy systems for the scattering functions, X- and Y-functions in the theory of radiative transfer [cf. "Radiative Transfer," by Chandrasekhar, Oxford University Press, 1950; Bellman, Kagiwada, Kalaba, and Ueno, *J. Math. Phys.* **9**, 906 (1968)]. It is of interest to mention that Eqs. (34) and (37) are readily converted into a system of mixed ordinary differential equations with prescribed initial values for the J, $\bar{J}$, J_r, and $\bar{J}_r$ functions.

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